Section 6: Duality, Sensitivity Analysis and the Simples Tableau

A. Duality

Def.6.1. The given LP whose dual we wish to find is called the PRIMAL LP.

2. symmetric form:

a.

Primal LP	**Dual LP**
min $z = \underline{c} \cdot \underline{x}$	max $z = \underline{\lambda} \cdot \underline{b}$
st	st
$A\underline{x} \geq \underline{b}$	$\underline{\lambda} A \leq \underline{c}$
$\underline{x} \geq \underline{0}$	$\underline{\lambda} \geq \underline{0}$

b.

Primal LP	**Dual LP**
min $z = \underline{c} \cdot \underline{x}$	max $z = \underline{\lambda} \cdot \underline{b}$
st	st
$A\underline{x} = \underline{b}$	$\underline{\lambda} A \leq \underline{c}$
$\underline{x} \geq \underline{0}$	$\underline{\lambda}$ unrestricted in sign

We will show how (b) follows from (a):

Rewrite the primal as

Primal LP

$$\begin{array}{ll} \min & z = \underline{c}\cdot\underline{x} \\ \text{st} & \\ & \mathbf{A}\underline{x} \ge \underline{b} \\ & -\mathbf{A}\underline{x} \ge -\underline{b} \\ & \underline{x} \ge \underline{0} \end{array}$$

which is of the form (a) only with

coefficient matrix $\left[\begin{array}{c} \mathbf{A} \\ \hdashline -\mathbf{A} \end{array}\right]$. Using $\left[\begin{array}{c;{2pt/2pt}c} \underline{u} & \underline{v} \end{array}\right]$ as our dual vector which is partitioned, we get the dual problem is

Dual LP

$$\begin{array}{ll} \max & z = \underline{u}\cdot\underline{b} - \underline{v}\cdot\underline{b} \\ \text{st} & \\ & \underline{u}\mathbf{A} - \underline{v}\mathbf{A} \le \underline{c} \\ & \underline{u} \ge \underline{0} \\ & \underline{v} \ge \underline{0} \end{array}$$

Let $\underline{\lambda} = \underline{u} - \underline{v}$. Then the dual LP

Dual LP

becomes

$$\begin{aligned} \max \quad & z = \underline{\lambda}\,\underline{\mathbf{b}} \\ \text{st} \quad & \underline{\lambda}\mathbf{A} \le \underline{\mathbf{c}} \\ & \underline{\lambda} \text{ unrestricted in sign} \end{aligned}$$

c.

Primal LP	**Dual LP**
$\max \quad z = \underline{\mathbf{c}} \cdot \underline{\mathbf{x}}$	$\min \quad z = \underline{\lambda} \cdot \underline{\mathbf{b}}$
st	st
$\mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{b}}$	$\underline{\lambda}\mathbf{A} \ge \underline{\mathbf{c}}$
$\underline{\mathbf{x}} \ge \underline{\mathbf{0}}$	$\underline{\lambda}$ unrestricted in sign

(c) can be obtained from (b) but this will be of your next assignment.

Primal Variables

$$\begin{array}{c|cccccc|cc}
 & & & & \downarrow & & & & \\
 & x_1 & x_2 & \cdots & x_j & \cdots & x_n & & \\
\hline
 & c_1 & c_2 & \cdots & c_j & \cdots & c_n & & \\
\hline
 & a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 & y_1 \\
3. & a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 & y_2 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m & y_m \\
 & \uparrow & & & \text{jth dual constr} & & & &
\end{array}$$

We now look at the dual of the Reddy Mikks Co. problem.

Recall (in std form): $\min z = -3x_E - 2x_I$

s.t.

$$x_E + 2x_I + s_1 = 6 \quad (1)$$

$$2x_E + x_I + s_2 = 8 \quad (2)$$

$$-x_E + x_I + s_3 = 1 \quad (3)$$

$$x_I + s_4 = 2 \quad (4)$$

$x_E \geq 0,\ x_I \geq 0,\ s_i \geq 0\ i=1,\ldots,4$

Primal Variables

x_E	x_I	s_1	↓ s_2	s_3	s_4	b	
–3	–2	0	0	0	0	0	
1	2	1	0	0	0	6	y_1
2	1	0	1	0	0	8	y_2
–1	1	0	0	1	0	1	y_3
0	1	0	0	0	1	2	y_4
↑			**jth dual constr**				

Dual LP

$$\begin{array}{lllllll}
\text{Max} & w = 6y_1 & +8y_2 & +y_3 & +2y_4 & & \\
\text{st} & & & & & & \\
& y_1 & +2y_2 & -y_3 & & \le -3 & (1) \\
& 2y_1 & +y_2 & +y_3 & +y_4 & \le -2 & (2) \\
& y_1 & & & & \le 0 & (3) \\
& & y_2 & & & \le 0 & (4) \\
& & & y_3 & & \le 0 & (5) \\
& & & & y_4 & \le 0 & (6)
\end{array}$$

y_i unrestricted in sign $i = 1,\ldots,4$

Note that y_i "unrestricted in sign" does not really make sense given the constraints (3)-(6) of the dual so we have to rewrite this as

$$\begin{array}{lllllll}
\text{Max} & w = 6y_1 & +8y_2 & +y_3 & +2y_4 & & \\
\text{st} & & & & & & \\
 & y_1 & +2y_2 & -y_3 & & \leq & -3 \quad (1) \\
 & 2y_1 & +y_2 & +y_3 & +y_4 & \leq & -2 \quad (2) \\
y_i & \leq & 0 & i = 1,\ldots,4 & & &
\end{array}$$

In other words, the "unrestricted in sign is removed and (3)-(6) are replaced by the nonpositivity conditions.

We often want to be able to drop the labels "primal and dual" so we need a table to help us create a dual from any LP.

Rules for constructing the dual problem		
Max Problem		Min Problem
constraints		*variables*
$\geq$	$\Leftrightarrow$	$\leq$ **0**
$\leq$	$\Leftrightarrow$	$\geq$ **0**
$=$	$\Leftrightarrow$	**unrestricted**
variables		*constraints*
$\geq$ **0**	$\Leftrightarrow$	$\geq$
$\leq$ **0**	$\Leftrightarrow$	$\leq$
unrestricted	$\Leftrightarrow$	$=$

Example 6.2: Consider the LP problem

$$
\begin{array}{llllll}
\text{Max} & z = 2x_1 & +7x_2 & +3x_3 & & \\
\text{s.t.} & & & & & \\
& x_1 & +9x_2 & +x_3 & +x_4 & = 11 \quad (1) \\
& 2x_1 & -x_2 & +13x_3 & & = 21 \quad (2) \\
& x_i \geq 0 \quad i = 1, \ldots 4 & & & &
\end{array}
$$

Find its dual LP.

Solution: Write the tableau

Primal Variables			↓		
x_1	x_2	x_3	x_4	b	
-2	-7	-3	0		
1	9	1	1	11	y_1
2	-1	13	0	21	y_2
↑			jth dual constr		

Dual LP

$$\begin{array}{llcll}
\text{Max} & w & = & 11y_1 & +21y_2 \\
\text{s.t.} & & & & \\
y_1 & +2y_2 & \leq & -2 & (1) \\
9y_1 & -y_2 & \leq & -7 & (2) \\
y_1 & +13y_2 & \leq & -3 & (3) \\
y_1 & & \leq & 0 & (4) \\
y_i & \text{unrestricetd} & \text{in sign} & i = 1,2 &
\end{array}$$

We will first make the RHS nonnegative and change the Max to min

$$\begin{array}{llcll}
\text{Min} & w & = & -11y_1 & -21y_2 \\
\text{s.t.} & & & & \\
-y_1 & -2y_2 & \geq & 2 & (1) \\
-9y_1 & +y_2 & \geq & 7 & (2) \\
-y_1 & -13y_2 & \geq & 3 & (3) \\
-y_1 & & \geq & 0 & (4) \\
y_i & \multicolumn{4}{l}{\text{unrestricetd in sign } \; i = 1,2}
\end{array}$$

Let $y_i^* = -y_i$ $i = 1,2,3$. We also recognize that (4) is really a nonnegativity condition on y_1 since this variable is not really unrestricted in sign. We also write the problem in terms of the variables y_i instead of y_i^* since it is just a variable name.

We then have the dual problem as:

$$\begin{array}{llcll}
\text{Min} & w & = & 11y_1 & +21y_2 \\
\text{s.t.} & & & & \\
y_1 & +2y_2 & \geq & 2 & (1) \\
9y_1 & -y_2 & \geq & 7 & (2) \\
y_1 & +13y_2 & \geq & 3 & (3)
\end{array}$$

$y_1 \geq 0$, y_2 unrestricetd in sign

Lemma 6.1: Given the primal in standard form, then the value of the objective function of the primal at any feasible solution is greater than or equal to the value of the dual at any feasible solution.

Proof: Let $\underline{x}$ be a feasible solution to the primal in standard form and let $\underline{\lambda}$ be a solution to the dual problem.

$$z = \underline{c} \cdot \underline{x}$$

But

$\underline{\lambda}A \leq \underline{c}$ and since $\underline{x} \geq \underline{0}$, $\underline{c} \cdot \underline{x} \geq (\underline{\lambda}A) \cdot \underline{x} = \underline{\lambda} \cdot (A\underline{x}) = \underline{\lambda} \cdot \underline{b} = w$

and so the lemma is proved.

Theorem 6.2: (Optimality Criterion) If $\underline{x}_*$ is feasible for the primal $\underline{\lambda}_*$ is a feasible solution for the dual such that $\underline{c} \cdot \underline{x}_* = \underline{\lambda}_* \cdot \underline{b}$, then $\underline{x}_*$ and $\underline{\lambda}_*$ are optimal feasible solutions to the primal and dual problems respectively.

Proof: Let $\underline{\lambda}$ be a feasible solution for the dual. Then, by the lemma,

$$\underline{c} \cdot \underline{x}_* \geq \underline{\lambda} \cdot \underline{b}$$

But, by hypothesis, $\underline{c} \cdot \underline{x}_* = \underline{\lambda}_* \cdot \underline{b}$ so $\underline{\lambda}_* \cdot \underline{b} \geq \underline{\lambda} \cdot \underline{b} \Rightarrow \underline{\lambda}_*$ is optimal feasible. A similar argument shows that $\underline{x}_*$ is optimal feasible.

Theorem 6.3: (Duality Theorem) If the primal and dual problems both have feasible solutions, then they both have optimal feasible solutions and the objective function values at these optimal solutions are equal. Furthermore, if one of the problems has an unbounded solution, then the other is infeasible.

Proof: (1) Suppose the dual has a feasible solution $\underline{\lambda}$. Then by Lemma 6.1, $\underline{\lambda} \cdot \underline{b}$ is a lower bound on the objective function of the primal so the primal cannot be unbounded. Similarly, if the primal has a feasible solution $\underline{x}$, then, by Lemma 6.1, $\underline{c} \cdot \underline{x}$ is an upper bound for the dual so the dual cannot be unbounded.

(2) Suppose the primal has an obfs . Let B_o be the optimal basis and B_o^{-1} the inverse of the optimal basis.

Claim: $\underline{\lambda} = \underline{c}_{B_o} B_o^{-1}$ is optimal feasible for the dual.

Proof: Since B_o^{-1} i.s the optimal basis, then the vector of reduced costs is nonnegative; i.e.

$$\underline{r} = \underline{c} - \underline{c}_{B_o} B_o^{-1} A \geq \underline{0}$$

i.e. $\lambda A \leq \underline{c}$ so that λ is feasible for the dual.

Now $\lambda \cdot \underline{b} = c_{B_o} B_o^{-1} \underline{b} = \underline{c}_{B_o} \cdot x_{B_o} = \underline{c} \cdot \underline{x}$ where $\underline{x}$ is the obfs associate with the optimal basis B_o. Applying the Optimality Criterion Theorem, completes the proof of the claim which concludes the proof of the theorem.

Example 6.3: Find the dual of the LP

$$\begin{array}{llllllll}
\max & z & = & 5x_1 & + & 12x_2 & + & 4x_3 \\
\text{st} & & & & & & & \\
x_1 & + & 2x_2 & + & x_3 & \leq & 10 & (1) \\
2x_1 & - & x_2 & + & 3x_2 & = & 8 & (2) \\
x_i & \geq & 0 & i = & 1,2,3 & & &
\end{array}$$

Solution: The LP in standard form is

$$\min \quad z = -5x_1 - 12x_2 - 4x_3$$

st

$$\begin{array}{lllllll}
x_1 + 2x_2 + x_3 + s_1 & = & 10 & (1) \\
2x_1 - x_2 + 3x_2 & = & 8 & (2) \\
x_i \geq 0 \quad i = 1,2,3 \quad s_1 \geq 0 & & &
\end{array}$$

Dual:

$$\max \quad w = 10y_1 + 8y_2$$

st

$$\begin{array}{llll}
y_1 + 2y_2 & \leq & -5 & (1) \\
2y_1 - y_2 & \leq & -12 & (2) \\
y_1 + 3y_2 & \leq & -4 & (3) \\
y_1 & \leq & 0 & (4)
\end{array}$$

y_1, y_2 unrestricted in sign

This is equivalent to

$$\begin{array}{lllll} \min & w = -10y_1 - 8y_2 & & & \\ \text{st} & & & & \\ -y_1 - 2y_2 & \geq & 5 & (1) \\ -2y_1 + y_2 & \geq & 12 & (2) \\ -y_1 - 3y_2 & \geq & 4 & (3) \\ -y_1 & \geq & 0 & (4) \\ y_1, y_2 & \text{unrestricted in sign} & & \end{array}$$

Replacing y_1 by $-y_1$ and y_2 by $-y_2$ gives the dual as

$$
\begin{array}{lllll}
\min & w = & 10y_1 + 8y_2 & & \\
\text{st} & & & & \\
y_1 + 2y_2 & \geq & 5 & & (1) \\
2y_1 - y_2 & \geq & 12 & & (2) \\
y_1 + 3y_2 & \geq & 4 & & (3) \\
y_1 \geq 0, & y_2 & \text{unrestricted in sign} & &
\end{array}
$$

We refer to the original as the primal LP and the dual as the dual LP. Solve the primal LP by the simplex method.

Basic	x_1	x_2	x_3	s_1	r_2	Solution	Phase 1
$-z$	0	0	0	0	1	0	
s_1	1	2	1	1	0	10	
r_2	2	-1	3	0	1	8	
$-z$	-2	1	-3	0	0	-8	"0"
s_1	1	2	1	1	0	10	$\frac{10}{1} = 10$
r_2	2	-1	3	0	1	8	$\frac{8}{3}$
$-z$	0	0	0	0	1	0	end of Phase 1
s_1	$\frac{1}{3}$	$\frac{7}{3}$	0	1	$-\frac{1}{3}$	$\frac{22}{3}$	
x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{8}{3}$	

Phase 2 with r_1 carried along:

$-z$	-5	-12	-4	0	0	0	**Phase 2**
s_1	$\frac{1}{3}$	$\frac{7}{3}$	0	1	$-\frac{1}{3}$	$\frac{22}{3}$	
x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{8}{3}$	
$-z$	$-\frac{7}{3}$	$-\frac{40}{3}$	0	0	$\frac{4}{3}$	$\frac{32}{3}$	**"0"**
s_1	$\frac{1}{3}$	$\frac{7}{3}$	0	1	$-\frac{1}{3}$	$\frac{22}{3}$	
x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{8}{3}$	
$-z$	$-\frac{3}{7}$	0	0	$\frac{40}{7}$	$-\frac{4}{7}$	$\frac{368}{7}$	**"1"**
x_2	$\frac{1}{7}$	1	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{22}{7}$	$\frac{22/7}{1/7} = 22$
x_3	$\frac{5}{7}$	0	1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{26}{7}$	$\frac{26/7}{5/7} = \frac{26}{5}$

$-z$	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5}$	$\frac{274}{5}$	optimal
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$	
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$	

The solution to the original primal problem is

$$\boxed{\begin{aligned} z &= 54\tfrac{4}{5} \\ x_1 &= \frac{26}{5} \\ x_2 &= \frac{12}{5} \\ x_3 &= 0 \end{aligned}}$$

Check: $5 \times \frac{26}{5} + 12 \times \frac{12}{5} = \frac{130}{5} + \frac{144}{5} = \frac{274}{5} = 54\frac{4}{5}$

Claim: dual solution is

$$\boxed{\begin{aligned} w &= 54\tfrac{4}{5} \\ y_1 &= -\frac{29}{5} \\ y_2 &= \frac{2}{5} \end{aligned}}$$

Recall: Primal

$$\begin{aligned} \min \quad & z = \underline{c} \cdot x \\ \text{st} \quad & \\ & Ax = \underline{b} \\ & x \geq \underline{0} \end{aligned}$$

Dual

$$\begin{aligned} \max \quad & w = \underline{\lambda} \cdot \underline{b} \\ \text{st} \quad & \\ & \underline{\lambda} A \leq \underline{c} \\ & \underline{\lambda} \ \text{unrestricted in sign} \end{aligned}$$

Partition $A = \left[\begin{array}{c:c} B & D \end{array} \right]$ and $\underline{c} = \left[\begin{array}{c:c} \underline{c}_B & \underline{c}_D \end{array} \right]$ and recall that the reduced costs are $\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D$.

Claim: $\lambda_* = \underline{c}_B B^{-1}$ is an optimal feasible solution of the dual problem where B is the optimal basis of the primal.

Pf: First note that

$$\lambda_* A = \underline{c}_B B^{-1} A = \underline{c}_B B^{-1} \left[\begin{array}{c:c} B & D \end{array} \right] = \left[\begin{array}{c:c} \underline{c}_B B^{-1} B & \underline{c}_B B^{-1} D \end{array} \right]$$

$$= \left[\begin{array}{c:c} \underline{c}_B & \underline{c}_B B^{-1} D \end{array} \right] \leq \left[\begin{array}{c:c} \underline{c}_B & \underline{c}_D \end{array} \right]$$

Note that the last inequality holds since at the optimal tableau,

$\underline{r} \geq \underline{0}$ and since $\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D \Rightarrow \underline{c}_D \geq \underline{c}_B B^{-1} D$.

Therefore $\lambda_* = \underline{c}_B B^{-1}$ is feasible.

To show that it is optimal, we will compute $\underline{\lambda}_*\underline{b}$ and show that it is equal to $\underline{c}_B\underline{x}_B$ where $\underline{x}_B = B^{-1}\underline{b}$ is obfs for the primal.

$\underline{\lambda}_*\underline{b} = \underline{c}_B B^{-1}\underline{b} = \underline{c}_B\underline{x}_B = \underline{c}\underline{x}$ so by the Optimality Theorem, $\underline{\lambda}_*$ is optimal.

<u>Question:</u> **Why are the coefficients on the primal tableau under the slacks in the z-row the optimal feasible solution to the dual?**

<u>Answer:</u> **Look at the example. In the optimal tableau the basic variables are x_2 and x_1. Looking at the original tableau,**

$B = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. What is B^{-1}? $B^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$

Check: $$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} + \frac{1}{5} & 2 \times -\frac{1}{5} + \frac{2}{5} \\ -\frac{2}{5} + 2 \times \frac{1}{5} & (-1) \times \frac{-1}{5} + 2 \times \frac{2}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

What is $\underline{c}_B B^{-1}$? Since B^{-1} updates the matrix, then by the pivoting operation it is $\begin{pmatrix} -\frac{29}{5} & \frac{2}{5} \end{pmatrix}$.

Check: $$\begin{pmatrix} -12 & -5 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{-29}{5} & \frac{2}{5} \end{pmatrix}.$$

<u>Solution to the dual LP:</u> Recall that $\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D$. In our example

$$\mathbf{D} = \left(\begin{array}{ccc} x_3 & s_1 & r_2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{array}\right).$$

Note: $\underline{r}$ is generalized to include r_2 but not for simplex method. If a column of the identity matrix is not in D (i.e. the variable is basic), then

$$c_i - \underline{c}_B B^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = r_i \text{ (of basic)} = 0 \text{ so } \underline{c}_B B^{-1}\left(\ \ \right) = c_i$$

Look at part of D that corresponds to the identity matrix (in our case the columns s_1 and r_2 so

$$\underline{c}_B B^{-1} I = \underline{c}_B B^{-1} = \underline{\lambda}^*$$

and so $\underline{\lambda}^*$ can be obtained from the objective function row of the optimal tableau as follows:

Step 1: Look at the objective function values in the optimal tableau corresponding to the columns of the I matrix in the initial tableau.

Step 2: Look at the original cost coefficients corresponding to those variables (i.e. in our example, corresponding to s_1 and r_2. These are 0 since our initial cost function is

$$z = -5x_1 - 12x_2 - 4x_3 + 0s_1 + 0r_2$$

Remark: In most cases, the setup is such that the coefficients obtained in Step 2 are 0.

Step 3: Sol to dual = Step 2 – Step 1 (In our example,

$$\boxed{\begin{aligned} y_1 &= 0 - \frac{29}{5} = -\frac{29}{5} \\ y_2 &= 0 - \frac{-2}{5} = \frac{2}{5} \end{aligned}}$$

This corresponds to $\lambda^* = \left(-\frac{29}{5}, \ \frac{2}{5} \right)$

We should get the same result by taking $\underline{c}_B B^{-1}$. In our case

$$\underline{c}_B B^{-1} = \left(\begin{array}{ccc} & -12 & -5 \\ x_2 & \leftarrow \text{cost coeff} \rightarrow & x_1 \end{array}\right)\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} -\frac{29}{5} & \frac{2}{5} \end{pmatrix}$$

Note: We have the solution for the dual LP

$$\begin{array}{llllll}
\min & w & = & -10y_1 & - & 8y_2 \\
\text{st} & & & & & \\
-y_1 & - & 2y_2 & \geq & 5 & (1) \\
-2y_1 & + & y_2 & \geq & 12 & (2) \\
-y_1 & - & 3y_2 & \geq & 4 & (3) \\
-y_1 & & & \geq & 0 & (4) \\
y_1, y_2 & \text{unrestricted} & \text{in sign} & & &
\end{array}$$

To get the solution to the dual LP form

$$\begin{array}{llllll}
\min & w & = & 10y_1 & + & 8y_2 \\
\text{st} & & & & & \\
y_1 & + & 2y_2 & \geq & 5 & (1) \\
2y_1 & - & y_2 & \geq & 12 & (2) \\
y_1 & + & 3y_2 & \geq & 4 & (3) \\
y_1 \geq & 0, & y_2 & \multicolumn{3}{l}{\text{unrestricted in sign}}
\end{array}$$

we have to take $y_1^{'} = -y_1$ and $y_2^{'} = -y_2$ which yields the solution

$$\boxed{\begin{array}{l} y_1 = \dfrac{29}{5} \\ y_2 = -\dfrac{2}{5} \\ w = 54\dfrac{4}{5} \end{array}} .$$

Example 6.4: Consider the LP

$$
\begin{array}{lllllll}
\min & z & = & -x_1 & - & 4x_2 & - & 3x_3 \\
\text{st} \\
2x_1 & + & 2x_2 & + & x_3 & & \le 4 \quad (1) \\
x_1 & + & 2x_2 & + & 2x_3 & & \le 6 \quad (2) \\
x_i & \ge & 0 & i = 1,2,3
\end{array}
$$

In standard form:

$$
\begin{array}{lllllllll}
\min & z & = & -x_1 & - & 4x_2 & - & 3x_3 \\
\text{st} \\
2x_1 & + & 2x_2 & + & x_3 & + & s_1 & & = 4 \quad (1) \\
x_1 & + & 2x_2 & + & 2x_3 & & & + \; s_2 & = 6 \quad (2) \\
x_i & \ge & 0 & i = 1,2,3 & s_i & \ge & 0 & i = 1,2
\end{array}
$$

Basic	x_1	x_2	x_3	s_1	s_2	Solution	
$-z$	-1	-4	-3	0	0	0	"0"
s_1	2	2	1	1	0	4	$\frac{4}{2}=2$
s_2	1	2	2	0	1	6	$\frac{6}{2}=3$
$-z$	3	0	-1	2	0	8	"1"
x_2	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2	$\frac{2}{\frac{1}{2}}=4$
s_2	-1	0	1	-1	1	2	$\frac{2}{1}=2$
$-z$	2	0	0	1	1	10	optimal
x_2	$\frac{3}{2}$	1	0	1	$-\frac{1}{2}$	1	
x_3	-1	0	1	-1	1	2	

At iteration "0": basic variables s_1, s_2 $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$

At iteration "1": basic variables x_2, s_2 $B = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ (from tableau "0") $B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix}$ (from tableau "1")

At optimal iteration: basic variables x_2, x_3 $B = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ (from tableau "0") $B^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix}$ (from optimal tableau)

At iteration 1: $c_B = \begin{pmatrix} -4 & 0 \end{pmatrix}$ original coefficients of x_2, s_2

Step 1: Obj. function coefficients at tableau 1 under variables which have the identity matrix in the initial tableau s_1, s_2

2,0

Step 2: As above except original cost coefficients

0,0

Step 3: $0-2=-2 \quad 0-0=0$

Alternatively, $c_B B^{-1} = \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \end{pmatrix}$

(–2,0) are called the SIMPLEX MLTIPLIERS. They correspond to an infeasible solution of the problem but can be thought of as synthetic prices.

Question: What is the dual?

Answer:

$$\begin{array}{lllllll} \max & w & = & 4y_1 & + & 6y_2 & \\ \text{st} & & & & & & \\ 2y_1 & + & y_2 & \leq & -1 & (1) & \\ 2y_1 & + & 2y_2 & \leq & -4 & (2) & \\ y_1 & + & 2y_2 & \leq & -3 & (3) & \\ y_1 & \leq & 0 & y_2 & \leq & 0 & \end{array}$$

Note: $y_1 = -2, y_2 = 0$ is infeasible as (3) is not satisfied.

Remark: Look at the difference between the RHS and LHS of the dual constraints: at any iteration

$-1 - (2y_1 + y_2)$ (1) z-coefficient of x_1

$-4 - (2y_1 + 2y_2)$ (2) z-coefficient of x_2

$-3 - (y_1 + 2y_2)$ (3) z-coefficient of x_3

$0 - y_1$ (4) z-coefficient of s_1

$0 - y_2$ (5) z-coefficient of s_2

e.g. at iteration "1", $y_1 = -2, y_2 = 0$

$$-1 - (2\times(-2) + 0) = -1 + 4 = 3 = \text{z-coeff of } x_1$$

$$-4 - (2\times(-2) + 2\times 0) = -4 + 4 = 0 = \text{z-coeff of } x_2$$

$$-3 - (-2 + 2\times 0) = -3 + 2 = -1 = \text{z-coeff of } x_3$$

$$0 - (-2) = 0 + 2 = 2 = \text{z-coeff of } s_1$$

$$0 - 0 = 0 = \text{z-coeff of } s_2$$

At optimal iteration: basic variables x_2, x_3

$B = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ **from orig. tab.** $B^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix}$ **from final tab.**

$$\underline{c}_B = \begin{pmatrix} -4 & -3 \end{pmatrix} \Rightarrow \underline{\lambda}^* = \underline{c}_B B^{-1} = \begin{pmatrix} -4 & -3 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \end{pmatrix}$$

Note that $\underline{\lambda}^* = \begin{pmatrix} -1 & -1 \end{pmatrix}$ is feasible for the dual:

$$\begin{array}{lllllll}
2(-1) & + & -1 & = & -1 & \leq & -1 \quad (1) \\
2(-1) & + & 2(-1) & = & -4 & \leq & -4 \quad (2) \\
-1 & +2(-1) & & = & -3 & \leq & -3 \quad (3) \\
-1 & \leq & 0, & -1 & \leq & 0 &
\end{array}$$

Note that the three constraints are binding when the rank is only 2. On constraint must be redundant. On examining (1)-(3), we note that (1) is redundant since (2) gives

$y_1 + y_2 \leq -2$ so that $2y_1 + y_2 \leq -2 + y_1$ and since $y_1 \leq 0$ then

$2y_1 + y_2 \leq -2 \leq -1$ so (1)is redundant.

The solution to the dual could also have been obtained from the optimal tableau as follows:

Step 1: 1 1 coefficients in optimal tableau

Step 2: 0 0 original cost coefficients over I

Step 3: 0–1=–1 0–1=–1

2. primal-dual computation:

obj. funct. row = RHS dual constr. − LHS dual constr.

At optimal iteration:

$-1 \quad -(2(-1) + -1) = -1 + 3 = 2$ =z-coeff of x_1 (1)

$-4 \quad -(2(-1) + 2(-1)) = -4 + 4 = 0$ =z-coeff of x_2 (2)

$-3 \quad -(-1 + 2(-1)) = -3 + 3 = 0$ =z-coeff of x_3 (3)

$0 \quad -(-1) = 1$ =z-coeff of s_1 (4)

$0 \quad -(-1) = 1$ =z-coeff of s_2 (5)

END OF MATH 3171

(This section will be continued next term.)

A. Economic Interpretation of Duality

1. general setup

Consider the problem of maximizing profit subject to availability of resources (linear production model)

Primal obj. function: max $z = \sum_{j=1}^{n} c_j x_j = \text{PROFIT}$

such that $\begin{array}{c} \underline{A}\underline{x} \leq \underline{b} \\ \underline{x} \geq 0 \end{array}$.

Dual (after letting $y_i' = -y_i$) min $w = \sum_{i=1}^{m} b_i y_i$

such that $\begin{array}{c} \underline{y}\underline{A} \geq \underline{c} \\ \underline{y} \geq 0 \end{array}$.

a. objective function of dual = objective function of primal when optimality is reached

DIMENSIONAL ANALYSIS

$$\text{dollars (profit)} = \sum\left(\text{units of resource i}\right) \times {}^{\text{dollars}}\!\big/_{\text{unit resource i}}$$

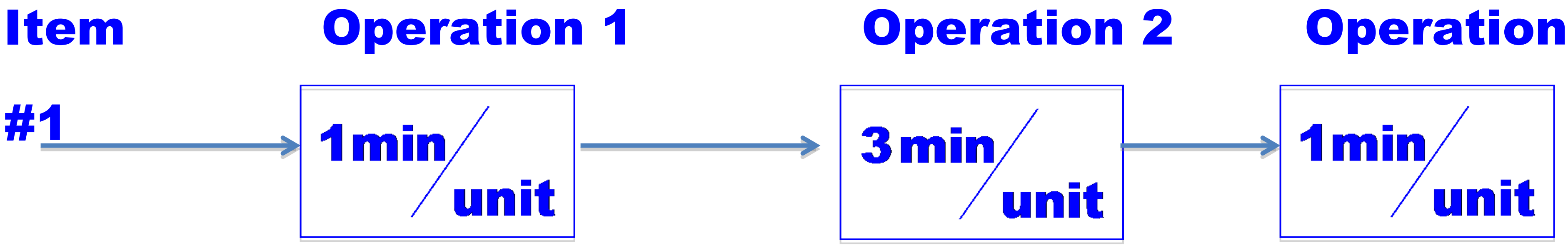
i.e. the dual variable represents the worth per unit of resource *i*.

<u>Definition 6.1:</u> The dual variables are called the SHADOW PRICES.

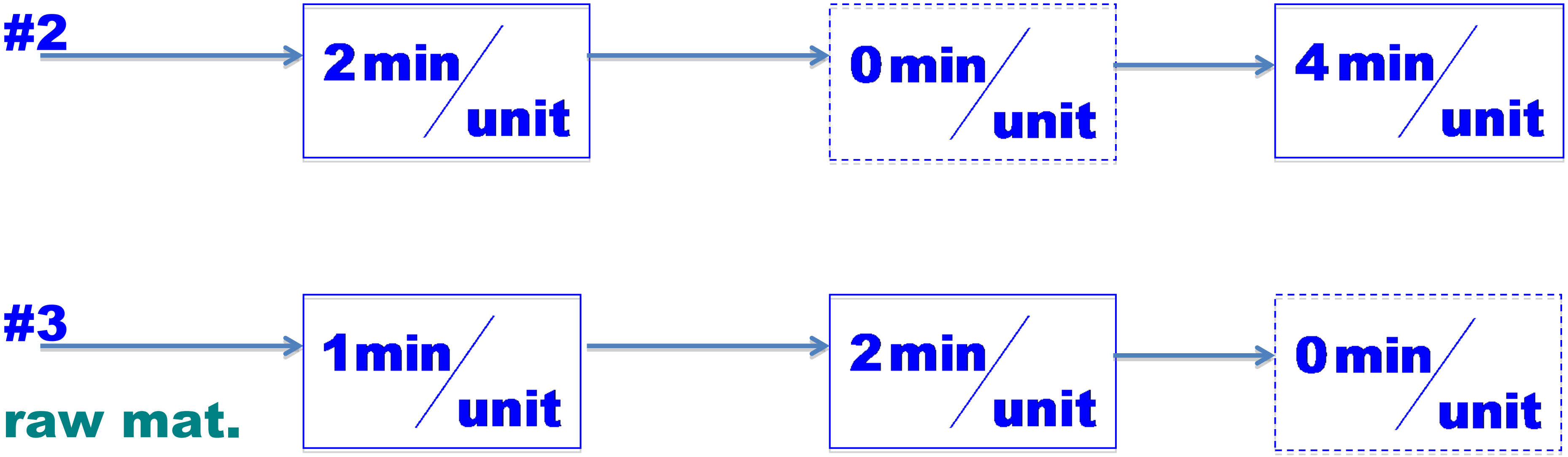
b. the unstable situation is PROFIT< WORTH OF RESOURCES

[<u>NOTE</u>: Putting the problem into standard form would make the unstable statement –PROFIT>–WORTH OF RESOURCES]

<u>**Example 6.2: (Application of Duality to Product-Mix Problem)**</u>



Item #1 → **Operation 1** 1min/unit → **Operation 2** 3min/unit → **Operation 3** 1min/unit



#2 → 2 min/unit → 0 min/unit → 4 min/unit

#3 → 1 min/unit → 2 min/unit → 0 min/unit

raw mat.

Max. usage: Operation 1: 430 min/day

Operation 2: 460 min/day

Operation 3: 420 min/day

Per item profit: Item #1: \$3.00 Item # 2: \$2.00 Item #3: \$5.00

The above is an input-ouput view of linear production model problem.)

Let x_i = # of items of type i produced daily i = 1, 2, 3

LP problem becomes:

$$
\begin{array}{llllllll}
\text{Max} & z & = & 3x_1 + 2x_2 & + & 5x_3 & & \\
\text{st} & & & & & & & \\
x_1 & + & 2x_2 & + & x_3 & \leq & 430 & (1) \\
3x_1 & + & 0x_2 & + & 2x_3 & \leq & 460 & (2) \\
x_1 & + & 4x_2 & + & 0x_3 & \leq & 420 & (3) \\
x_i & \geq & 0 & i & = & 1,2,3 & &
\end{array}
$$

In standard form:

$$\min \quad z = -3x_1 - 2x_2 - 5x_3$$

$$\text{st}$$

$$\begin{array}{lllllllll}
x_1 & + & 2x_2 & + & x_3 & + & s_1 & & = 430 \quad (1) \\
3x_1 & + & 0x_2 & + & 2x_3 & & +s_2 & & = 460 \quad (2) \\
x_1 & + & 4x_2 & + & 0x_3 & & & +s_3 & = 420 \quad (3)
\end{array}$$

$$x_i \geq 0, \quad i = 1,2,3 \quad s_j \geq 0, \quad j = 1,2,3$$

Dual problem:

$$
\begin{array}{lllllll}
\max & w & = & 430y_1 & + & 460y_2 & + \; 420y_3 \\
\text{st} & & & & & & \\
y_1 & + & 3y_2 & + & y_3 & \leq & -3 \quad (1) \\
2y_1 & & & + & 4y_3 & \leq & -2 \quad (2) \\
y_1 & + & 2y_2 & & & \leq & -5 \quad (3) \\
y_1 & & & & & \leq & 0 \quad (4) \\
 & & y_2 & & & \leq & 0 \quad (5) \\
 & & & & y_3 & \leq & 0 \quad (6) \\
y_i & \text{unrestricted} & \text{in sign} & i = & 1,2,3 & &
\end{array}
$$

which becomes

$$\max \quad w = 430y_1 + 460y_2 + 420y_3$$

st

$$y_1 + 3y_2 + y_3 \leq -3 \qquad (1)$$

$$2y_1 + 4y_3 \leq -2 \qquad (2)$$

$$y_1 + 2y_2 \leq -5 \qquad (3)$$

$$y_i \leq 0 \qquad i = 1,2,3$$

The optimal tableau for the primal problem is

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
$-z$	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
s_3	2	0	0	–2	1	1	20

which yields the optimal solution is $\boxed{\begin{array}{c} z = 1350 \\ x_1 = 0 \\ x_2 = 100 \\ x_3 = 230 \end{array}}$ for the original max primal problem ($s_1 = 0, s_2 = 0, s_3 = 20$) Note that product 1 will not be produced in the optimal solution to the product-mix problem.

Dual Variables:

$$y_1 = 0 - 1 = -1$$
$$y_2 = 0 - 2 = -2$$

To go baack to the original primal which was *max*, we should convert dual to *min* problem. It becomes

$$\begin{array}{llllllll}
\min & w & = & -430y_1 & - & 460y_2 & - & 420y_3 \\
\text{st} & & & & & & & \\
y_1 & + & 3y_2 & + & y_3 & \leq & -3 & (1) \\
2y_1 & & & + & 4y_3 & \leq & -2 & (2) \\
y_1 & + & 2y_2 & & & \leq & -5 & (3) \\
y_i & \leq 0 & & i = & 1,2,3 & & &
\end{array}$$

Letting $y_i^{'} = -y_i \quad i = 1,2,3$ and dropping the ' since it is only a formality, we get

$$
\begin{array}{lllllllll}
\min & w & = & 430y_1 & + & 460y_2 & + & 420y_3 & \\
\text{st} & & & & & & & & \\
y_1 & + & 3y_2 & + & y_3 & \geq & 3 & (1) & \\
2y_1 & & & + & 4y_3 & \geq & 2 & (2) & \\
y_1 & + & 2y_2 & & & \geq & 5 & (3) & \\
y_i & \geq 0 & & i = & 1,2,3 & & & &
\end{array}
$$

The optimal solution is

$y'_1 = -y_1 = -(-1) = +1$, $y'_2 = -(-2) = +2$ and $y'_3 = -(0) = 0$ or in terms

of the y_i i=1,2,3, $\boxed{\begin{array}{l} y_1 = 1 \\ y_2 = 2 \\ y_3 = 0 \end{array}}$.

<u>Question:</u> **Product 1 is not used in the optimal product mix because it is not profitable to produce product 1** (i.e. $z_1 > c_1$)

How can product 1 be made profitable to produce?

Solution: Reduce usage of operations 1-3 in production of product 1, if possible.

Reducing the usage of operation 3 would not help since the marginal worth of operation 3 in the final tableau is 0. It is preferable to reduce the use of operation 2 since its shadow price is higher than the shadow price of operation 1.

Let $r_2 =$ reduction in minutes/unit of product 1 by operation 2.

Then $3 - r_2$ = new # of minutes of operation 2 to produce product 1.

To bring x_1 into the basis at the final tableau, let us first compute the new reduced cost for x_1 .

imputed cost (synthetic) of product 1 =$z_1 = y_1 + (3 - r_2)y_2 + y_3$.

At the optimal tableau, $y_1 = -1$, $y_2 = -2$, $y_3 = 0$ so

$z_1 = -1 + (3 - r_2)(-2) + 0 = -1 - 6 + 2r_2 + 0 = -7 + 2r_2$. For x_1 to enter (and product 1 to be profitable) we must have $c_1 - z_1 \le 0$; i.e.

$(-3 = c_1)$ so $-3 - (-7 + 2r_1) \le 0 \Rightarrow 2r_2 \ge 4 \Rightarrow r_2 \ge 2$

Hence if we reduce the usage of operation 1 fro 3 min/unit to 1 min/unit, it becomes profitable to produce product 1. (Note: If the reduce cost equals 0, we can have that variable enter the basis without affecting the value of the objective function.)

B. Dual Simplex Algorithm

Recall: reduced costs $\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D$. When the primal is non-optimal, $\underline{r} \not\geq \underline{0}$; i.e. there exists at least one component of $\underline{r}$ which is less than 0.

Letting $\underline{y} = \underline{c}_B B^{-1}$, we get $\underline{r} = \underline{c}_D - \underline{y} D$.

Hence, for some *j*, $\underline{y}\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} > c_j$ and this implies that $\underline{y}$ is infeasible for the dual; i.e. the simplex multipliers are feasible for the dual LP only at the optimal tableau.

1. Method:

a. Start with the primal tableau with basic solution which is infeasible but optimal (or better than optimal) in the sense that the dual solution solutions (i.e. the simplex multipliers) are feasible for the dual. (This implies that all reduced costs must be nonnegative.) We will move towards feasibility of the primal, while maintaining optimality.

Def. 6.3: a basic (not necessarily feasible) solution $x_B = B^{-1}\underline{b}$ of the primal problem is called **DUAL FEASIBLE** iff $\underline{y} = \underline{c}_B B^{-1}$ (the simplex multipliers) are feasible for the dual LP.

3. algorithm for the dual simplex method

STEP 1: (a) If x_B is dual feasible and $x_B \geq \underline{0}$, **STOP**. x_B is optimal.

(b) If $x_B \ngeq \underline{0}$, let $I = \left\{ i : \left(x_B \right)_i < 0 \right\}$.

STEP 2: If $\underline{a}_i \geq \underline{0}$, for any $i \in I$, **STOP** – the dual has no maximum so the primal is infeasible.

STEP 3: (a) Choose i^* so that $\left(x_B \right)_{i^*}$ is the most negative element of x_B; i.e. $\left(x_B \right)_{i^*} \leq \left(x_B \right)_i$ for all $1 \leq i \leq m$. This is the **EXITING CONDITION.** It is also called the **FEASIBILITY CONITION FOR THE PRIMAL.**

(b) Choose j* such that

$\frac{-r_{j^*}}{a_{i^*j^*}} = \min\left\{\frac{-r_j}{a_{i^*j}} : a_{i^*j} < 0\right\}$. This is the **ENTERING CONDITION.** It is also called the **OPTIMALITY CONDITION.** It maintains DUAL FEASIBILITY.

STEP 4: Go to Step 1.

Note: $\geq$ inequalities are a natural case to use dual simplex method instead of simplex method.

Example 6.4: Consider the LP problem

$$\min \quad z = 3x_1 + 4x_2 + 5x_3$$

st

$$x_1 + 2x_2 + 3x_3 \geq 5 \quad (1)$$

$$2x_1 + 2x_2 + x_3 \geq 6 \quad (2)$$

$$x_i \geq 0 \quad i = 1,2,3$$

In standard form, the LP becomes:

$$\begin{array}{lllllllllll}
\min & z & = & 3x_1 & + & 4x_2 & + & 5x_3 & & & \\
\text{st} & & & & & & & & & & \\
x_1 & + & 2x_2 & + & 3x_3 & - & s_1 & & = & 5 & (1) \\
2x_1 & + & 2x_2 & + & x_3 & & & - \; s_2 & = & 6 & (2)
\end{array}$$

$$x_i \geq 0 \quad i = 1,2,3 \quad s_i \geq 0 \quad i = 1,2$$

In (non-standard form)

$$\begin{array}{lllllllllll}
\min & z & = & 3x_1 & + & 4x_2 & + & 5x_3 & & & \\
\text{st} & & & & & & & & & & \\
-x_1 & - & 2x_2 & - & 3x_3 & + & s_1 & & = & -5 & (1) \\
-2x_1 & - & 2x_2 & - & x_3 & & & + \; s_2 & = & -6 & (2)
\end{array}$$

$$x_i \geq 0 \quad i = 1,2,3 \quad s_i \geq 0 \quad i = 1,2$$

Here there is an obvious basic but infeasible solution.

Dual Simplex Method

Basic	x_1	x_2	x_3	s_1	s_2	Solution	
$-z$	3	4	5	0	0	0	"0"
s_1	-1	-2	-3	1	0	-5	
s_2	-2	-2	-1	0	1	-6	
$-z$	0	1	$\frac{7}{2}$	0	$\frac{3}{2}$	-9	"1"
s_1	0	-1	$-\frac{5}{2}$	1	$-\frac{1}{2}$	-2	
x_1	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	3	
$-z$	0	0	1	1	1	-11	optimal
x_2	0	1	$\frac{5}{2}$	-1	$\frac{1}{2}$	2	
x_1	1	0	-2	1	-1	1	

Hence the optimal solution to the primal problem is $\boxed{\begin{array}{l} z = 11 \\ x_2 = 2 \\ x_1 = 1 \\ x_3 = 0 \end{array}}$.

Note: The advantage of this approach is it is faster because it does not require either the two phase method or the big M method or any artificial variables.

Sometimes we obtain an infeasible basic solution because of a change in the constraints after the problem has been solved.

Example 6.4: (Reddy Mikks Co.)

$$
\begin{array}{llll}
\text{Maximize} & z = 3x_E + 2x_I & & \\
\text{st} & & & \\
& x_E + 2x_I & \le 6 & (1) \\
& 2x_E + x_I & \le 8 & (2) \\
& -x_E + x_I & \le 1 & (3) \\
& x_I & \le 2 & (4) \\
& x_E \ge 0, \quad x_I \ge 0 & &
\end{array}
$$

In standard form:

$$
\begin{array}{llllllllll}
\text{Minimize} & z & = & -3x_E & - & 2x_I & & & & \\
\text{st} & & & & & & & & & \\
x_E & + & 2x_I & +s_1 & & & & = & 6 & (1) \\
2x_E & + & x_I & & +s_2 & & & = & 8 & (2) \\
-x_E & + & x_I & & & +s_3 & & = & 1 & (3) \\
 & & x_I & & & & +s_4 & = & 2 & (4)
\end{array}
$$

$$x_E \geq 0, \quad x_I \geq 0 \quad s_i \geq 0 \quad i = 1,\ldots,4$$

which has optimal tableau

Basic	x_E	x_I	s_1	s_2	s_3	s_4	$-z$	Solution
$-z$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	1	$+12\,{}^{2}\!/_{3}$
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$
s_3	0	0	-1	1	1	0	0	3
s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$

Suppose the availability of resource (1) is increased to 10 tons per day and the availability of resource (2) is decreased to 4 tons per day. The current basis at the optimal tableau is

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse of this basis is

$$\mathbf{B}^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix}$$

. Hence

$$\mathbf{B}^{-1}\underline{\mathbf{b}} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{16}{3} \\ -\frac{2}{3} \\ -5 \\ -\frac{10}{3} \end{pmatrix}$$

$\underline{\mathbf{c}}_{\mathbf{B}} = (-2 \quad -3 \quad 0 \quad 0)$ **so**

$$\underline{\mathbf{c}}_{\mathbf{B}}\mathbf{B}^{-1}\underline{\mathbf{b}} = \begin{pmatrix} -2 & -3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{16}{3} \\ -\frac{2}{3} \\ -5 \\ -\frac{10}{3} \end{pmatrix} = -\frac{26}{3}$$

so the updated last

tableau (formerly optimal tableau) becomes

Basic	x_E	x_I	s_1	s_2	s_3	s_4	$-z$	Solution
$-z$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	1	$+26/3$
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{16}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$-\frac{2}{3}$
s_3	0	0	-1	1	1	0	0	-5
s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$-\frac{10}{3}$

which is infeasible.

This is a perfect situation to apply Dual Simplex Algorithm.

Basic	x_E	x_I	s_1	s_2	s_3	s_4	$-z$	Solution	
$-z$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	1	$+26/3$	
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{16}{3}$	
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$-\frac{2}{3}$	
s_3	0	0	-1	1	1	0	0	-5	
s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$-\frac{10}{3}$	
$-z$	0	0	0	$\frac{5}{3}$	$\frac{1}{3}$	0	1	7	**optimal**
x_I	0	1	0	$\frac{1}{3}$	$\frac{2}{3}$	0	0	2	
x_E	1	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	1	
s_1	0	0	1	-1	-1	0	0	5	
s_4	0	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$	1	0	0	

Hence the solution to the new max LP problem is

$$\boxed{\begin{array}{c} z = 7000 \\ x_I = 2 \\ x_E = 1 \end{array}} .$$

C. Complementary Slackness Theorem

Theorem 6.5: At the optimal tableau

(1) $r_j > 0 \Rightarrow x_j = 0$ (i.e. $r_j x_j = 0$) where n_j is the surplus variable of the jth dual constraint

(2) $y_i < 0 \Rightarrow s_i = 0$ (i.e. $s_i y_i = 0$) .

Proof: (1) If, at the optimal tableau, $r_j > 0$, then x_j must be non-basic since the tableau is in canonical form. Hence $x_j = 0$.

(**Note:** r_j corresponds to the *surplus variable* of the *jth* dual constraint.)

(2) From the simplex algorithm, we know that the reduced cost coefficient corresponding to the slack s_i of the optimal tableau is $c_i - y_i$ where c_i is the orginal cost of s_i, so $c_i = 0$.

Therefore $0 - y_i = -y_i$ is the reduced cost coefficient. Since $y_i < 0$, then $r_i > 0 \Rightarrow s_i$ is non-basic $\Rightarrow s_i = 0$ so that $s_i y_i = 0$

Example 6.6: Primal LP:

$$
\begin{array}{llllllll}
\text{Max} & z & = & 5x_1 & -18x_2 & -6x_3 & +x_4 & \\
\text{st} & & & & & & & \\
2x_1 & & -x_3 & +x_4 & & & \leq & 20 \quad (1) \\
 & x_2 & -2x_3 & -x_4 & & & \leq & 30 \quad (2) \\
-3x_1 & +6x_2 & +3x_3 & +4x_4 & & & \leq & 24 \quad (3) \\
\end{array}
$$

$x_i \geq 0 \quad i = 1, \ldots, 4$

In standard form:

$$\begin{array}{lllllllll}
\text{Min} & z & = & -5x_1 & +18x_2 & +6x_3 & -x_4 & & \\
\text{st} & & & & & & & & \\
2x_1 & & -x_3 & +x_4 & +s_1 & & & = 20 & (1) \\
& x_2 & -2x_3 & -x_4 & & +s_2 & & = 30 & (2) \\
-3x_1 & +6x_2 & +3x_3 & +4x_4 & & & +s_3 & = 24 & (3)
\end{array}$$

$$x_i \geq 0 \quad i = 1, \ldots, 4 \quad s_i \geq 0 \quad i = 1, 2, 3$$

Dual LP:

$$\begin{array}{llllll}
\text{Max} & w & = & 20y_1 & +30y_2 & +24y_3 \\
\text{st} & & & & & \\
2y_1 & & -3y_3 & \le & 5 & (1) \\
 & y_2 & +6y_3 & \le & -18 & (2) \\
-y_1 & -2y_2 & +3y_3 & \le & -6 & (3) \\
3y_1 & -y_2 & +4y_3 & \le & 1 & (4) \\
y_i \le 0 & i = 1,2,3 & & & &
\end{array}$$

Optimal solution of primal:

$$\boxed{\begin{array}{c}
z = -112 \\
x_1 = 10 \\
x_2 = 9 \\
x_3 = 0 \\
x_4 = 0
\end{array}}$$

Optimal solution of dual:

$$\boxed{\begin{matrix} w = -112 \\ y_1 = -2 \\ y_2 = 0 \\ y_3 = -3 \end{matrix}}$$

For optimal primal: LHS (1)=20=RHS SLACK=0 $y_1 < 0$

LHS (2)=9<30=RHS SLACK>0 $y_2 = 0$

LHS (3)=24=RHS SLACK=0 $y_3 < 0$

Similarly for dual. Hence we have

$$\left(\text{slack at } x^*\right)_i = 0 \quad \text{or} \quad \left(\text{corresp. dual } y^*\right)_i = 0$$

Another way of stating the Complementary Slackness Theorem follows:

Theorem 6.7: If the primal LP is written as

$$\begin{array}{llll} \text{Max} & z & = \underline{c}\cdot\underline{x} & \\ \text{st} & & & \\ A\underline{x} & +\underline{s} & = & \underline{b} \\ \underline{x}\ge\underline{0} & \underline{s}\ge\underline{0} & & \end{array}$$

so that the dual LP is

$$\begin{array}{llll} \text{Min} & w & = & \underline{y}\cdot\underline{b} \\ \text{st} & & & \\ \underline{y}A & -\underline{n} & = & \underline{c} \\ \underline{y}\ge\underline{0} & n\ge\underline{0} & & \end{array}$$

where $\underline{s}$ are the slacks in the primal problem and $\underline{n}$ are the surplus variables in the dual LP. Let $\underline{x}$ be feasible primal LP solution and $\underline{y}$ be a feasible solution for the dual LP. Then $\underline{x}$ is optimal for the primal and $\underline{y}$ is optimal for the dual if and only if

$$s_j y_j = 0 \quad j = 1,\ldots,m \quad (1)$$

$$n_j x_j = 0 \quad j = 1,\ldots,n \quad (2)$$

Proof: Suppose conditions (1) and (2) hold.

Because of the (feasibility) nonnegativity conditions, (1) is equivalent to

$\underline{s}\cdot\underline{y}=0$ while (2) is equivalent to $\mathbf{n}\cdot\mathbf{x}=0$. Now

$\mathbf{w}=\underline{y}\underline{b}=\underline{y}(A\mathbf{x}+\underline{s})=\underline{y}A\mathbf{x}+\underline{y}\underline{s}$ (3) while

$\mathbf{z}=\underline{c}\mathbf{x}=(\underline{y}A-\mathbf{n})\mathbf{x}=\underline{y}A\mathbf{x}-\mathbf{n}\mathbf{x}$ (4)

so if the complementary slackness conditions and feasibility holds then $\mathbf{w}=\underline{y}A\underline{x}=\mathbf{z}$ so by the Optimality Theorem, $\underline{x}$ is optimal or the primal LP and $\underline{y}$ is optimal for the dual.

Conversely, suppose $\underline{x}$ is optimal for the primal and $\underline{y}$ is optimal for the dual. The equations (3) and (4) still hold and because of optimality w=z so that

$$\underline{\underline{y}A\underline{x}+\underline{y}\underline{s}=\underline{y}A\underline{x}-\underline{n}\underline{x}}$$

and so $\underline{y}\underline{s}=-\underline{n}\underline{x}$. But $\mathbf{x}\geq\underline{0},\mathbf{n}\geq\underline{0}\Rightarrow\mathbf{n}\cdot\mathbf{x}\geq 0$ and

$\underline{y} \geq \underline{0}, \underline{s} \geq \underline{0} \Rightarrow \underline{y} \cdot \underline{s} \geq 0$ so we must have $\underline{y}\underline{s} = -\underline{n}\underline{x} = 0$ and therefore, by nonnegativity, (1) and (2) (complementary slackness) must be true.

Remark: Complementary slackness $\Leftrightarrow$ optimal solution

$\Leftrightarrow$ Optimality Theorem

Keep in mind $z = \underline{c}\underline{x} \geq \underline{y}A\underline{x} = \underline{y}\underline{b} = w$

Back to Sensitivity Analysis

Recall: Why do we want to perform sensitivity analysis?

Answer: The problem is changed by changing circumstances;

e.g. new information from marketing dept., new costs, new R&D, etc.

Question: What can happen to the current optimal solution?

Answer: It can (1) become infeasible, (2) become non-optimal, or (3) become infeasible and non-optimal. We will provide an algorithm for handling (1) or (2). (3) will be treated by special cases.

1. changes affecting feasibility

Recall: $x_B = B^{-1}\underline{b}$. For the Reddy Miks Co LP

$$B^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix}$$

. Let B the current basis be the optimal basis.

Note: If we do not permit changes in the usage requirements of the basic variables (at the optimal tableau), then the only way that feasibility can be affected is by changing the $\underline{b}$ or by adding a new constraint.

a. change in $\underline{b}$

Suppose (in R-M LP) $\underline{b} = \begin{pmatrix} 8 \\ 8 \\ 2 \\ 4 \end{pmatrix}$ **. Updatig the RHS at the**

optimal tableau gives $B^{-1}\underline{b} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{8}{3} \\ 2 \\ \frac{4}{3} \end{pmatrix}$ **.**

The tableau is still optimal feasible; however the solution is now

$x_E = \frac{8}{3}, x_I = \frac{8}{3}, (s_3 = 2, s_4 = \frac{4}{3}, s_1 = s_2 = 0)$ **and the optiaml objective function value is now** $3 \cdot \frac{8}{3} + 2 \cdot \frac{8}{3} = \frac{40}{3}$ **.**

Note: The new objective function value could have been calculated without finding the new $\underline{x}_B$ as follows:

$$\underline{y}b = \begin{pmatrix} -\frac{1}{3} & -\frac{4}{3} & 0 & 0 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \\ 2 \\ 4 \end{pmatrix} = -\frac{8}{3} - \frac{32}{3} = -\frac{40}{3}$$

However, this technique would only be acceptable if we KNOW that $\underline{x}_B \geq \underline{0}$ (we know we are still at the optimal tableau.)

Question: What happens if $x_B \ngeq \underline{0}$?

We already had an example of such a situation and we used dual simplex method to get rid of the infeasibility.

1b. addition of a new constraint

e.g. $x_E + x_I \leq 5$

This is an example of a constraint which is satisfied by the current optimal feasible solution $x_E = \frac{10}{3}, x_I = \frac{4}{3}$ since

$$\frac{10}{3} + \frac{4}{3} = \frac{14}{3} < 5$$

(If we check graphically, we note the new constraint is actually redundant.)

Note: **The addition of a new constraint cannot improve the value of the objective function; i.e. if the LP problems is**

$$\begin{aligned} \min \quad & z = \underline{c}x \\ \text{st} \quad & \\ & Ax = \underline{b} \\ & x \geq \underline{0} \end{aligned}$$

and if a new constraint is added, then the new optimal value of z $\geq$ curent optimal value of z (without new constraint).

Conclusion: **If the additional constraint is satisfied by the current optimal feasible solution, then the new constraint has no effect on the optimal feasible solution.**

Case 1: **Constraint is satisfied by optimal feasible solution with < or > but not equality.**

Then the new constraint is either non-binding or redundant.

Case 2: Constraint is satisfied by optimal feasible solution with equality occurring.

The new constraint is binding and the optima feasible solution is overconstrained and so a redundancy of some sort has been introduced.

Case 3: The additional constraint is NOT satisfied by the current optimal feasible solution.
The new constraint will become binding and we obtain the new optimal feasible solution by the Dual Simplex Algorithm.

To recover feasibility:

STEP 1: Put the constraint into equality form (standard form of LP) by using slack or surplus variables. If it is already an equality, introduce an artificial slack.

STEP 2: Put the new tableau into canonical form.

STEP 3: Apply Dual Simplex Algorithm.

e.g. $x_E \leq 3$ a new constraint for Reddy-Mikks LP

We note that at the optimal solution, $x_E = \frac{10}{3} \nleq 3$

$x_E + s_5 = 3$

Basic	x_E	x_I	s_1	s_2	s_3	s_4	s_5	$-z$	Solution
$-z$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	0	1	$+{}^{38}\!/_{3}$
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	0	$\frac{10}{3}$
s_3	0	0	-1	1	1	0	0	0	3
s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	0	$\frac{2}{3}$
s_5	1	0	0	0	0	0	1	0	3

Putting it into canonical form:

Basic	x_E	x_I	s_1	s_2	s_3	s_4	s_5	$-z$	Solution	
$-z$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	0	1	$+\,^{38}\!/_{3}$	
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	0	$\frac{4}{3}$	
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	0	$\frac{10}{3}$	
s_3	0	0	-1	1	1	0	0	0	3	
s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	0	$\frac{2}{3}$	
s_5	0	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	1	0	$-\frac{1}{3}$	
$-z$	0	0	1	0	0	0	2	1	12	op
x_I	0	1	0.5	0	0	0	-0.5	0	1.5	
x_E	1	0	0	0	0	0	1	0	3	
s_3	0	0	-0.5	0	1	0	1.5	0	2.5	
s_4	0	0	-0.5	0	0	1	0.5	0	0.5	
s_2	0	0	-0.5	1	0	0	-3.5	0	0.5	

Note that the optimal objective function value is 12.

2. changes affecting optimality

Question: **If we assume that our changes maintain feasibility, what changes can affect optimality?**

Answer: **Our answer is found in the primal-dual computations**

obj. function values = RHS of dual const. – LHS of dual const.

$\Rightarrow$ **optimality can only be affected by either**

(1) changes in the cost coefficients of the objective function

and/or

(2) changes in the usage of the resources in each activity

e.g. Reddy-Mikks

change in cost coefficients: **max** $z = 5x_E + 4x_I$

in standard form:min $z = -5x_E - 4x_I$

Method 1: Calculate $\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D$

$$\underline{c}_D = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad \underline{c}_B = \begin{pmatrix} -4 & -5 & 0 & 0 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\text{. Therefore}$$

$$B^{-1}D = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \\ -1 & 1 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \text{ so}$$

$$c_B B^{-1} D = \begin{pmatrix} -4 & -5 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \\ -1 & 1 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -1 & -2 \end{pmatrix}.$$ **Hence**

$\mathbf{r} = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$. **The other reduced costs over the basic variables are 0 so the tableau is still optimal.**

Method 2:

(1) If the change in the cost coefficients include a change in the cost of a basic variable, then

(a) determine the new dual values by $c_B B^{-1}$;

(b) compute the new z-coefficents by means of the primal-dual computations

RHS of dual const. –LHS of dual const.= obj row value

e.g. in this case, the change involves a basic variable so

$$\text{(a) } \underline{c}_B B^{-1} = \begin{pmatrix} -4 & -5 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 0 & 0 \end{pmatrix}$$

original costs over identity matrix: 0 0 0 0

Step 3: $0-(-1)=+1 \quad 0-(-2)=+2 \quad 0-0=0 \quad 0-0=0$

Therefore the tableau is still optimal.

Example 2: Reddy Mikks with $z=-7x_I-2x_E$.

$$\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} -2 & -7 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \\ -1 & 1 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

so $\underline{r} = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 4 \end{pmatrix}$. Furthermore

$$z_0 = -\underline{c}_B B^{-1} \underline{b} = -\begin{pmatrix} -2 & -7 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ \frac{10}{3} \\ 3 \\ \frac{2}{3} \end{pmatrix} = -\left(-\frac{78}{3}\right) = 26$$

In this case the new tableau is not optimal and becomes

Basic	x_E	x_I	s_1	s_2	s_3	s_4	$-z$	Solution
$-z$	0	0	-1	4	0	0	1	$\frac{38}{3}$
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$
s_3	0	0	-1	1	1	0	0	3
s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$

D. Change in Usage of Resources by an Activity

Recall: We are restricting changes to the coefficients of those variables which are non-basic t the optimal tableau. This ensures that we are not affecting feasibility.

Example 6.8 Suppose that the price of exterior paint is $4000/ton and the price of interior paint is #1000/ton.

(a) What is the new optimal solution?

(b) Suppose that the amount of raw material A use by the interior paint is increased to 4 tons/ton of interior paint while the amount of material B for interior paint is raised to 3 tons/ton of interior paint. What is the new optimal solution?

Solution: Solve R.M. problem with $z = -4x_E - x_I$

Step1: Take optimal tableau for R.M. problem (or solve new problem if the tableau is not given for old cost coefficents).

Basic	x_E	x_I	s_1	s_2	s_3	s_4	$-z$	Solution
$-z$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	1	$+{}^{38}\!/\!_{3}$
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$
s_3	0	0	-1	1	1	0	0	3
s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$

1a.: update the tableau

$$\underline{c}_B = \begin{pmatrix} -1 & -4 & 0 & 0 \end{pmatrix} \; B^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix}$$

Hence

$$\underline{r} = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} -1 & -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \\ -1 & 1 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} & -4 \end{pmatrix} \text{ so}$$

$$= \begin{pmatrix} -\frac{2}{3} & 4 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} -\frac{2}{3} & 4 \end{pmatrix} \qquad -z_0 = -[(-4)\cdot\frac{10}{3} + (-1)\cdot\frac{4}{3}] = \frac{44}{3}$$

(Alternately, we could have calculated $-z_0$ by using the formula $z = \underline{c}_B B^{-1} \underline{b} = \underline{y}\underline{b}$; i.e. The dual values are obtained by

<u>Step (i):</u> $-\frac{2}{3} \quad \frac{7}{3} \quad 0 \quad 0$ **(values in the z-row over I at optimal tableau)**

<u>Step (ii):</u> $0 \quad 0 \quad 0 \quad 0$ **(original cost coefficients corresponding to these variables)**

<u>Step (iii):</u> Step (ii) – Step (i)= $0-\frac{2}{3} \quad 0-\left(-\frac{7}{3}\right) \quad 0-0 \quad 0-0$

so the dual values are $-\frac{2}{3} \quad \frac{7}{3} \quad 0 \quad 0$ **so that**

$$\underline{y}\underline{b} = \begin{pmatrix} -\frac{2}{3} & \frac{7}{3} & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \\ 1 \\ 2 \end{pmatrix} = -\frac{12}{3} + \frac{56}{3} = \frac{44}{3}$$

although it would be silly to use this method since the other calculation is much easier.)

Basic	x_E	x_I	s_1	s_2	s_3	s_4	$-z$	Solution
$-z$	0	0	$-\frac{2}{3}$	4	0	0	1	$+{}^{44}\!/_{3}$
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$
s_3	0	0	-1	1	1	0	0	3
s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$
$-z$	0	$\frac{3}{2}$	0	$\frac{11}{3}$	0	0	1	$\frac{48}{3} = 16$ optimal
s_1	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	0	2
x_E	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	4
s_3	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	0	0	5
s_4	0	1	0	0	0	1	0	2

Step 2a: We note that x_I is non-basic in the optimal tableau. Hence a change in the usage of resources for interior paint will not affect our current B^{-1}.

Step 2: UPDATE THE TABLEAU.

The only column which would need updating is the x_I column.

Note that since B^{-1} does not change and since the costs were not changed, $c_B B^{-1}$ (the dual values) remain unchanged so that

$-z_0 = -\underline{c}_B B^{-1}\underline{b}$ is also unchanged.

e.g. $$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{3}{2} \\ \frac{5}{2} \\ 1 \end{pmatrix}$$

$$\uparrow$$

new x_l column

Step 3: Update the z-coeficients.

$\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D \quad \underline{c}_D = \begin{pmatrix} -1 & 0 \end{pmatrix} \quad \underline{c}_B = \begin{pmatrix} 0 & -4 & 0 & 0 \end{pmatrix}$ so

$$\underline{r} = \begin{pmatrix} -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 3 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \end{pmatrix} - \begin{pmatrix} -6 & -2 \end{pmatrix}$$

Hence our new reduced cost line is $\underset{\sim}{r} = \begin{pmatrix} 5 & 2 \end{pmatrix}$

Step 3a: **If** $\underset{\sim}{r} \geq \underset{\sim}{0}$ **STOP. Our current optimal solution is still optimal. (In this example we would have stopped.)**

Basic	x_E	x_I	s_1	s_2	s_3	s_4	$-z$	Solution
$-z$	0	5	0	2	0	0	1	$\frac{48}{3} = 16$
s_1	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	0	0	0	2
x_E	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	0	0	4
s_3	0	$\frac{5}{2}$	0	$\frac{1}{2}$	1	0	0	5
s_4	0	1	0	0	0	1	0	2

Otherwise

Step 4: Use Simplex Algorithm to obtain the optimal tableau.

Example 6.9: (Addition of a new activity) Suppose we wish to produce a new paint (e.g. cheap exterior paint) which uses ¾ of a ton of material A and ¾ of a ton of material B per ton of paint and which sells for $1500 per ton. What is our optimal solution?

Solution: Let x_C = amount of cheap exterior paint (in tons) produced.

Our LP problem is

$$\begin{array}{llllllll} \max & z & = & 3x_E & +2x_I & +\frac{3}{2}x_C & & \\ \text{st} & & & & & & & \\ x_E & +2x_I & +\frac{3}{4}x_C & & \le & 6 & (1) \\ 2x_E & +x_I & +\frac{3}{4}x_C & & \le & 8 & (2) \\ -x_E & +x_I & -x_C & & \le & 1 & (3) \\ & x_I & & & \le & 2 & (4) \end{array}$$

$$x_E \ge 0 \quad x_I \ge 0 \quad x_C \ge 0$$

Note: Marketing Dept. says

$1 +$ (total exterior paint) $\ge$ total interior paint so that

$1 + (x_E + x_C) \ge x_I$ from which (2) follows.

We could put this problem into standard form and solve "from scratch"; however, it is more expedient to use sensitivity analysis.

Consider x_C as if it had originally been in the model with usage coefficients $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ changing to $\begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ -1 \\ 0 \end{pmatrix}$. Place this column after the x_I column in the tableau.

<u>Step 1:</u> done

Note that x_C is non-basic in current optimal tableau.

<u>Step 2:</u> UPDATE THE Z-COEFICIENTS

$$\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D$$

$$= \begin{pmatrix} -\frac{3}{2} & 0 & 0 \end{pmatrix} - \begin{pmatrix} -2 & -3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{4}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & 1 & 0 \\ \frac{3}{4} & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{2} & 0 & 0 \end{pmatrix} - \begin{pmatrix} -2 & -3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{4} & -\frac{1}{3} & \frac{2}{3} \\ -1 & -1 & 1 \\ -\frac{1}{4} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{2} & 0 & 0 \end{pmatrix} - \begin{pmatrix} -\frac{5}{4} & -\frac{1}{3} & -\frac{4}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{3} & \frac{4}{3} \end{pmatrix}$$

Hence the tableau becomes

Basic	x_E	x_I	x_C	s_1	s_2	s_3	s_4	$-z$	Solution	
$-z$	0	0	$-\frac{1}{4}$	$-\frac{1}{3}$	$\frac{4}{3}$	0	0	1	$+{}^{38}\!/_{3}$	"0"
x_I	0	1	$\frac{1}{4}$	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$	
x_E	1	0	$\frac{1}{4}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$	
s_3	0	0	-1	-1	1	1	0	0	3	
s_4	0	0	$-\frac{1}{4}$	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$	
$-z$	0	1	0	1	1	0	0	1	14	optimal
x_C	0	4	1	$\frac{8}{3}$	$-\frac{4}{3}$	0	0	0	$\frac{16}{3}$	
x_E	1	-1	0	-1	1	0	0	0	2	
s_3	0	4	0	$\frac{5}{3}$	$-\frac{1}{3}$	1	0	0	$\frac{25}{3}$	
s_4	0	1	0	0	0	0	1	0	2	

END OF SECTION 6